Graphs, Heaps

Exam-Level 08

Announcements

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
			3/13 Mid-semester Survey Due		3/15 Lab 8 Due Project 2B/C Checkpoint and Design Doc Due TRS 3 (11-1pm)	
	3/18 Homework 3 Due			3/21 **Midterm 2**		

Content Review

Trees, Revisited (and Formally Defined)

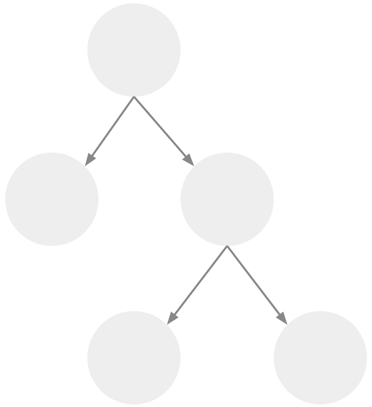
Trees are structures that follow a few basic rules:

1. If there are N nodes, there are N-1 edges
2. There is exactly 1 path from root to every other node
3. The above two rules means that trees are fully connected and contain no cycles

A parent node points towards its child.

The root of a tree is a node with no parent nodes.

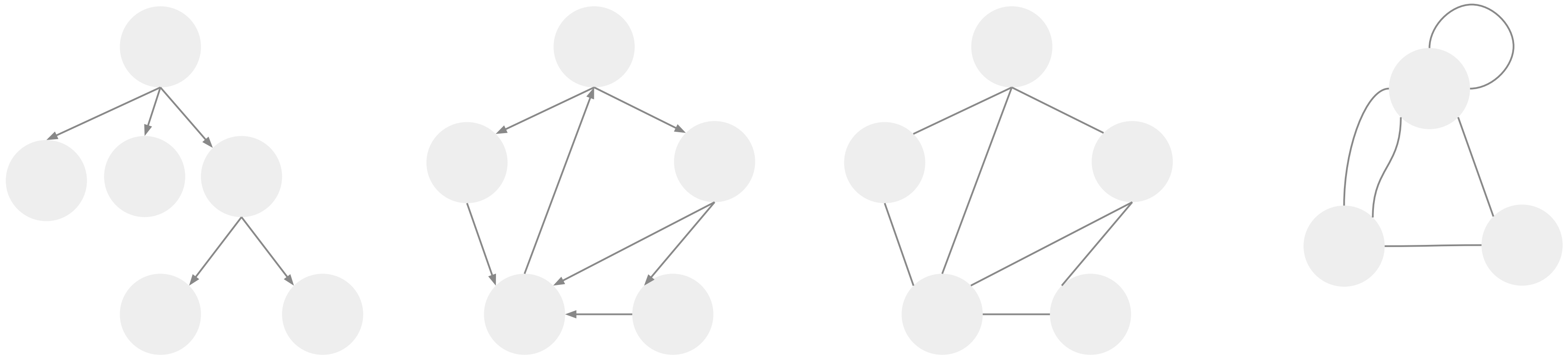
A leaf of a tree is a node with no child nodes.



Graphs

Trees are a specific kind of **graph,** which is more generally defined as below:

1. Graphs allow cycles
2. Simple graphs don't allow parallel edges (2 or more edges connecting the same two nodes) or self edges (an edge from a vertex to itself)
3. Graphs may be directed or undirected (arrows vs. no arrows on edges)

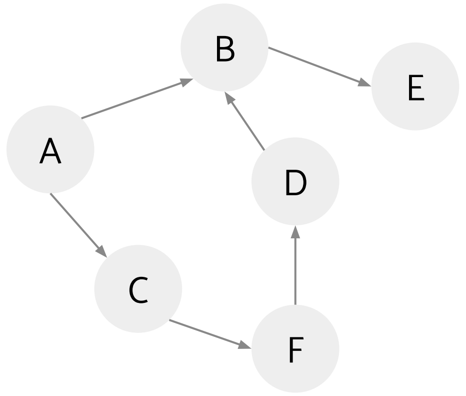


Check! How would you describe each of these graphs (in terms of directedness and cycles)?

Graph Representations

Adjacency lists list out all the nodes connected to each node in our graph:

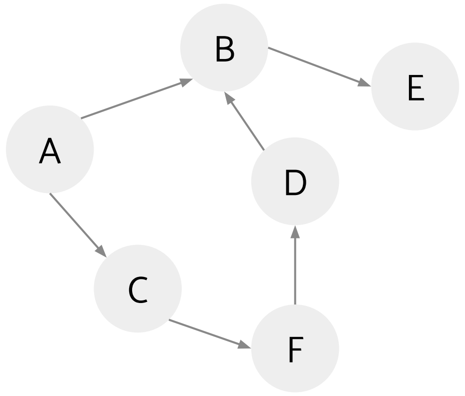
A	B , C
B	E
C	F
D	B
E	
F	D



Graph Representations

Adjacency matrices are true if there is a line going from node A to B and false otherwise.

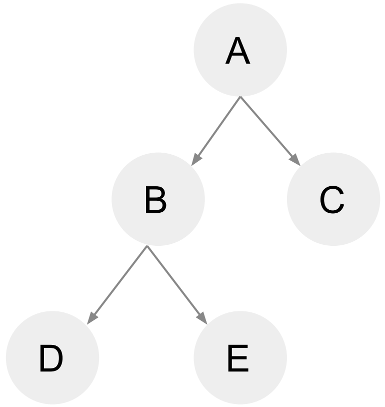
	A	B	C	D	E	F
A	0	1	1	0	0	0
B	0	0	0	0	1	0
C	0	0	0	0	0	1
D	0	1	0	0	0	0
E	0	0	0	0	0	0
F	0	0	0	1	0	0



Breadth First Search

Breadth first search means visiting nodes based off of their distance to the source, or starting point. For trees, this means visiting the nodes of a tree level by level. Breadth first search is one way of traversing a graph.

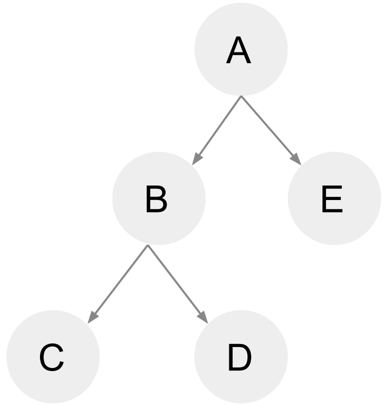
BFS is usually done using a queue.

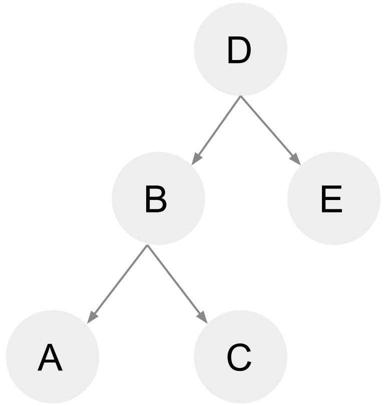


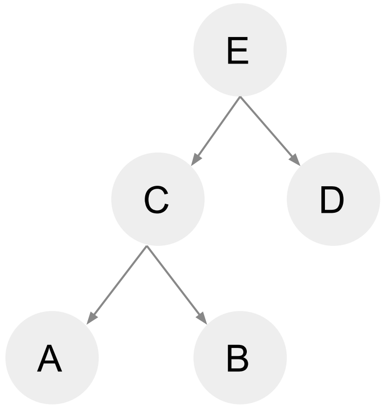
```
BFS(G):
     Add G.root to queue
     While queue not empty:
          Pop node from front of queue and visit
          for each immediate neighbor of node:
               Add neighbor to queue if not
               already visited
```

Depth First Search

Depth First Search means we visit each subtree (subgraph) in some order recursively. DFS is usually done using a stack. Note that for graphs more generally, it doesn't really make sense to do in-order traversals.







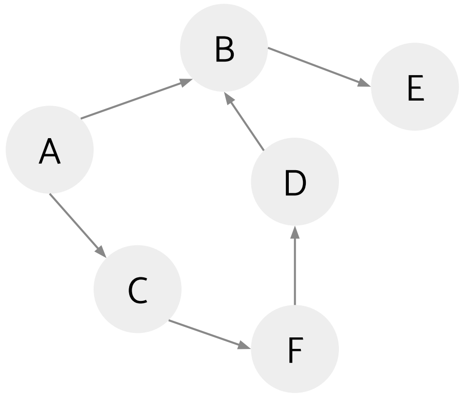
Pre-order traversals visit the parent node before visiting child nodes.*

In-order traversals visit the left child, then the parent, then the right child.

Post-order traversals visit the child nodes before visiting the parent nodes.*

* in binary trees, we visit the left child before right child

General Graph DFS Pseudocode (Stack)



```
DFS(start):
    stack = {start}, visited = {}
    while stack not empty:
        n = top node in stack
        visited.add(n), preorder.add(n)
        if n has unvisited neighbors:
            push n's next unvisited
            neighbor onto stack
        else:
            pop n off top of stack
            postorder.add(n)
    return preorder, postorder
```

Preorder: "Visit the node as soon as it enters the stack: myself, then all my children"

Postorder: "Visit the node as soon as it leaves the stack: all my children, then myself"

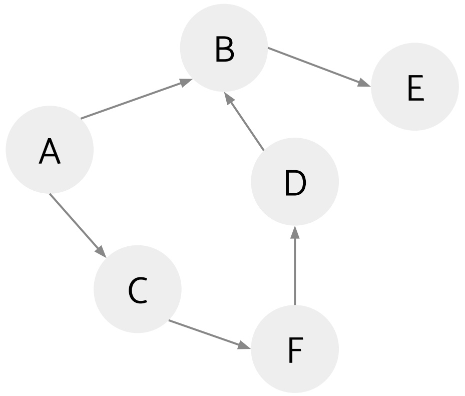
* in-order for binary trees:

```
DFSInorder(T):
    DFSInorder(T.left)
    visit T.root
    DFSInorder(T.right)
```

"Visit my left child, then myself, then my right child"*

* can be done with a stack, but usually easier with recursive

General Graph DFS Pseudocode (Recursive)



```
DFS(start):
    preorder.add(start)
    visited.add(start)
    for each neighbor of start:
        if neighbor not visited:
            DFS(neighbor)
    postorder.add(start)
    return preorder, postorder
```

Note: technically can add:

```
if start.neighbors is empty
  preorder.add(start)
  visited.add(start)
  postorder.add(start)
```

as base case, but the code on the left will skip the loop if neighbors is empty.

* in-order for binary trees:

```
DFSInorder(T):
    DFSInorder(T.left)
    visit T.root
    DFSInorder(T.right)
```

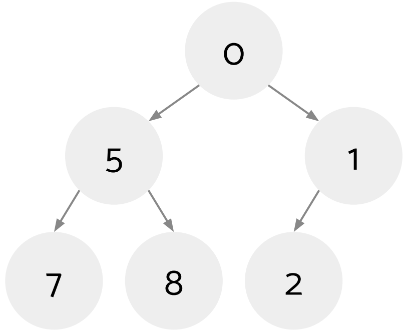
"Visit my left child, then myself, then my right child"*

* can be done with a stack, but usually easier with recursive

Heaps

Heaps are special trees that follow a few invariants:

1. Heaps are complete - the only empty parts of a heap are in the bottom row, to the right
2. In a min-heap, each node must be *smaller* than all of its child nodes. The opposite is true for max-heaps.

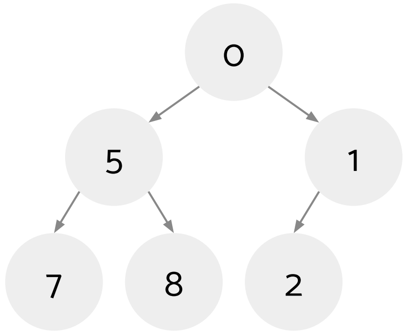


Check! What makes a binary min-heap different from a binary search tree?

Heap Representation

We can represent binary heaps as arrays with the following setup:

1. The root is stored at index 1 (not 0 - see points 2 and 3 for why)
2. The left child of a binary heap node at index i is stored at index 2i
3. The right child of a binary heap node at index i is stored at index 2i + 1

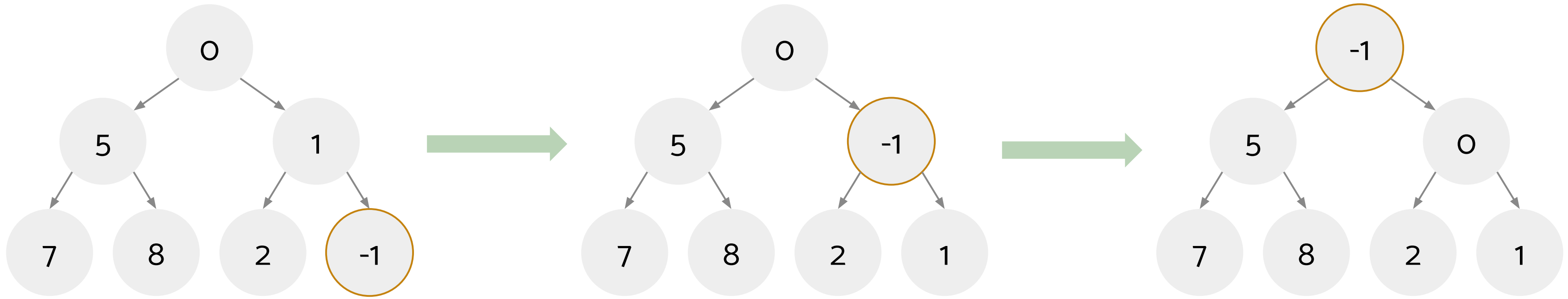


```
[-, 0, 5, 1, 7, 8, 2]
```

Check! What kind of graph traversal does the ordering of the elements in the array look like starting from the root at index 1?

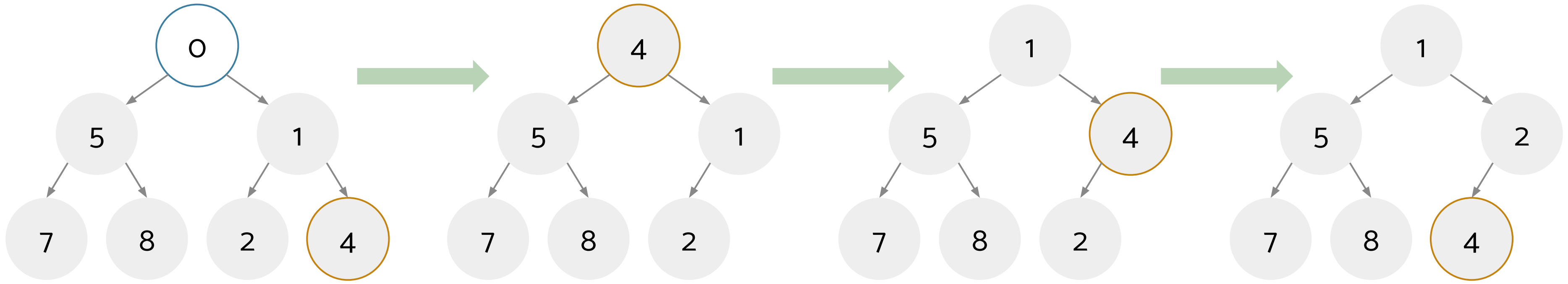
Insertion into (Min-)Heaps

We insert elements into the next available spot in the heap and bubble up as necessary: if a node is smaller than its parent, they will swap. (Check: what changes if this is a max heap?)



Root Deletion from (Min-)Heaps

We swap the last element with the root and bubble down as necessary: if a node is greater than its children, it will swap with the lesser of its children. (Check: what changes if this is a max heap?)



Heap Asymptotics (Worst case)

Operation	Runtime
insert	$\Theta(\log N)$
findMin	$\Theta(1)$
removeMin	$\Theta(\log N)$

Worksheet

1a Graph Conceptuals (T/F)

1. If a graph with n vertices has n – 1 edges, it must be a tree.
2. Every edge is looked at exactly twice in every iteration of DFS on a connected, undirected graph.
3. In BFS, let d(v) be the minimum number of edges between a vertex v and the start vertex. For any two vertices u, v in the fringe, |d(u) – d(v)| is always less than 2.

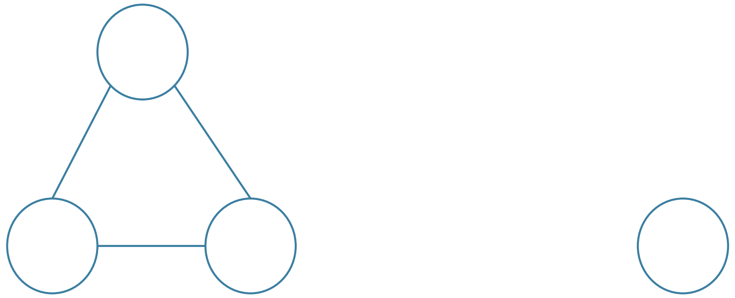
1a Graph Conceptuals (T/F)

1. If a graph with n vertices has n – 1 edges, it must be a tree.

1a Graph Conceptuals (T/F)

1. If a graph with n vertices has n – 1 edges, it must be a tree.

False. Could be disconnected.



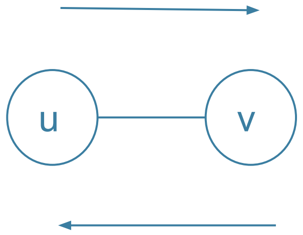
1a Graph Conceptuals (T/F)

2. Every edge is looked at exactly twice in each full run of DFS on a connected, undirected graph.

1a Graph Conceptuals (T/F)

2. Every edge is looked at exactly twice in each full run of DFS on a connected, undirected graph.

True. The two vertices the edge is connecting will look at that edge when it's their turn.



1a Graph Conceptuals (T/F)

3. In BFS, let d(v) be the minimum number of edges between a vertex v and the start vertex. For any two vertices u, v in the fringe (the fringe is a queue in BFS), |d(u) − d(v)| is always less than 2.

1a Graph Conceptuals (T/F)

3. In BFS, let d(v) be the minimum number of edges between a vertex v and the start vertex. For any two vertices u, v in the fringe (the fringe is a queue in BFS), |d(u) − d(v)| is always less than 2.

True.

```
[2, 2, 3, 3, 4]
```

added after dequeuing dist-3 node

1a Graph Conceptuals (T/F)

3. In BFS, let d(v) be the minimum number of edges between a vertex v and the start vertex. For any two vertices u, v in the fringe (the fringe is a queue in BFS), |d(u) – d(v)| is always less than 2.

True.

```
[2, 2, 3, 3, 4]
```

but can't deque dist-3 until all dist-2 nodes done!

added after dequeuing dist-3 node

1b Graph Conceptuals

Given an undirected graph, provide an algorithm that returns true if a cycle exists in the graph, and false otherwise. Also, provide a Θ bound for the worst case runtime of your algorithm.

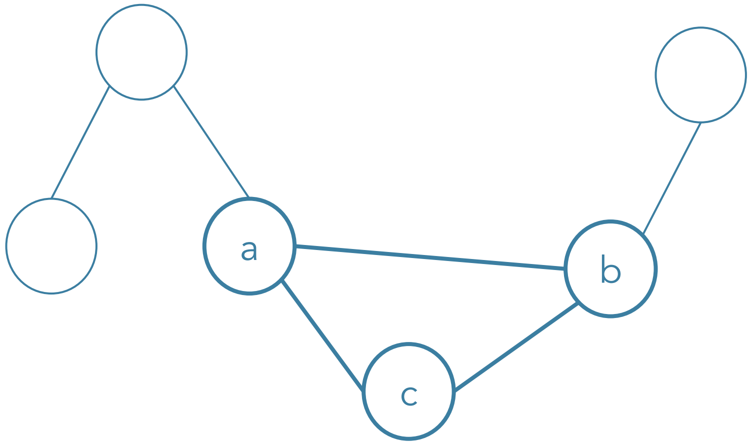
1b Graph Conceptuals

Given an undirected graph, provide an algorithm that returns true if a cycle exists in the graph, and false otherwise. Also, provide a Θ bound for the worst case runtime of your algorithm. You may use either an adjacency list or an adjacency matrix to represent your graph. (We are looking for an answer in plain English, not code).

Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.

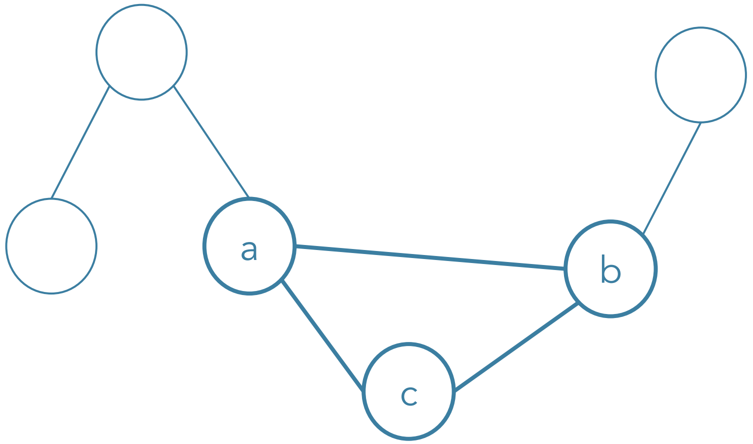
1b Graph Conceptuals

Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.



1b Graph Conceptuals

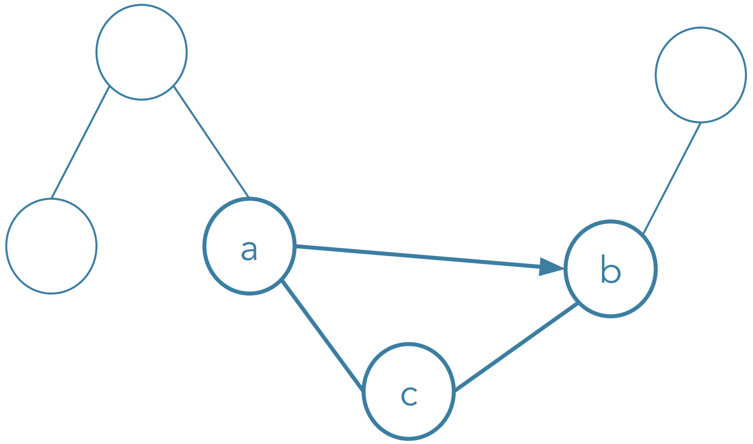
Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.



dfs(a)

1b Graph Conceptuals

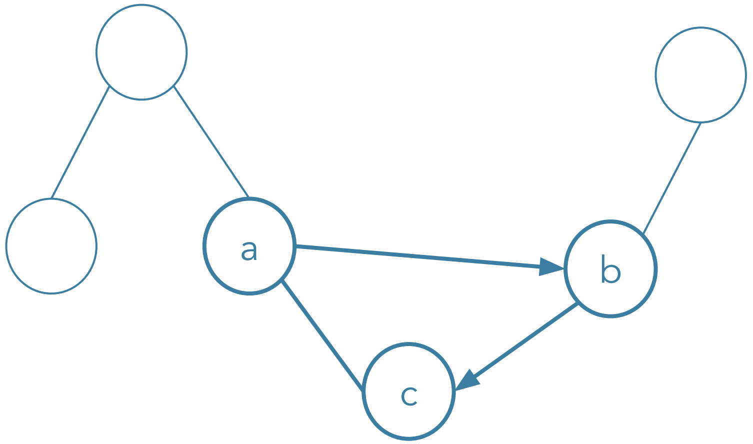
Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.



dfs(a) → dfs(b)

1b Graph Conceptuals

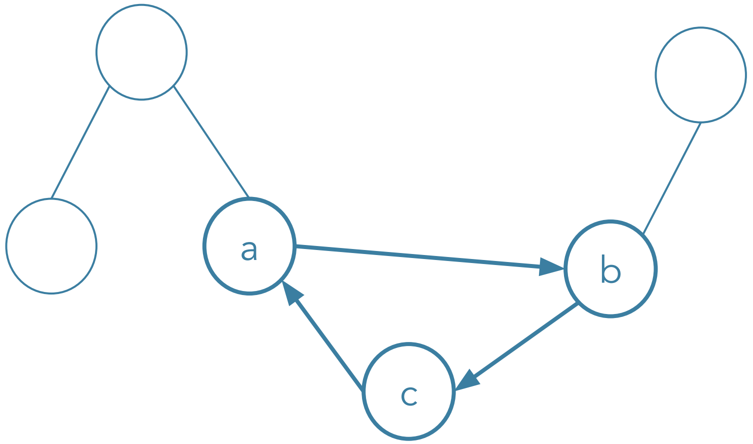
Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.



dfs(a) → dfs(b) → dfs(c)

1b Graph Conceptuals

Basic Idea: Keep track of visited nodes, do a DFS and if we visit any already visited nodes there is a cycle.



dfs(a) → dfs(b) →
dfs(c) → **dfs(a)**

repeat = cycle

2 Fill in the Blanks

1. `removeMin` has a best case runtime of ____ and a worst case runtime of _____.

2 Fill in the Blanks

1. `removeMin` has a best case runtime of Θ(1) and a worst case runtime of Θ(logN).

Best case: only one swap down is required, thus finishing in constant time
Worst case: sink down from top to the bottom. Height = Θ(logN)

2 Fill in the Blanks

2. `insert` has a best case runtime of ______ and a worst case runtime of ______.

2 Fill in the Blanks

2. `insert` has a best case runtime of Θ(1) and a worst case runtime of Θ(logN).

Best case: no bubbling up required

Worst case: bubble up from bottom to top. Height = Θ(logN)

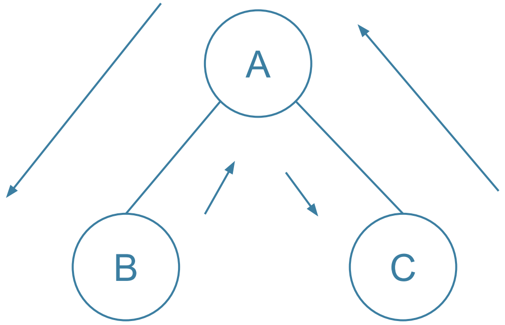
2 Fill in the Blanks

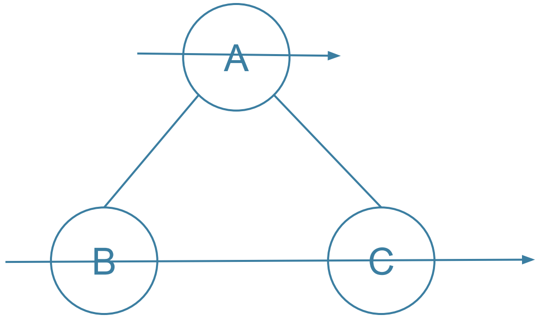
3. A ________ or ________ traversal on a min-heap *may* output the elements in sorted order. Assume there are at least 3 elements in the min-heap.

2 Fill in the Blanks

3. A **pre-order** or **level-order** traversal on a min-heap *may* output the elements in sorted order. Assume there are at least 3 elements in the min-heap.

Any traversal must output ***the top node first***. Only pre-order and level-order obey this constraint.





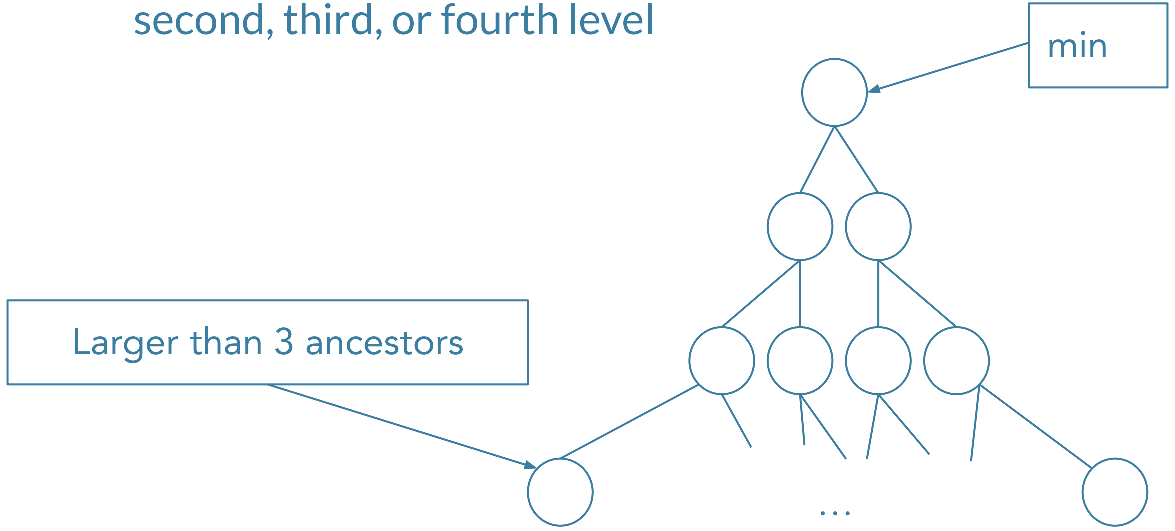
2 Fill in the Blanks

4. The fourth smallest element in a min-heap with 1000 elements can appear in ____ places in the heap.

2 Fill in the Blanks

4. The fourth smallest element in a min-heap with 1000 elements can appear in 14 places in the heap.

second, third, or fourth level



2 Fill in the Blanks

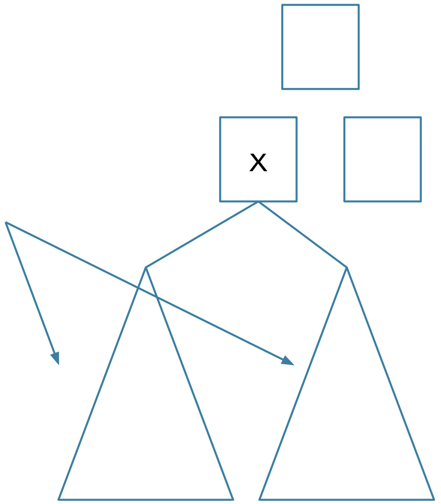
5. Given a min-heap with 2^n - 1 elements, for an element to be on the second level it must be less than _____ element(s) and greater than __ element(s).

2 Fill in the Blanks

5. Given a min-heap with $2^n - 1$ elements, for an element to be on the second level it must be less than $2^{(N-1)} - 2$ element(s) and greater than 1 element(s).

must be greater than the topmost and less than the elements in its subtree

minus the top node, take only the left half, and then remove the node x



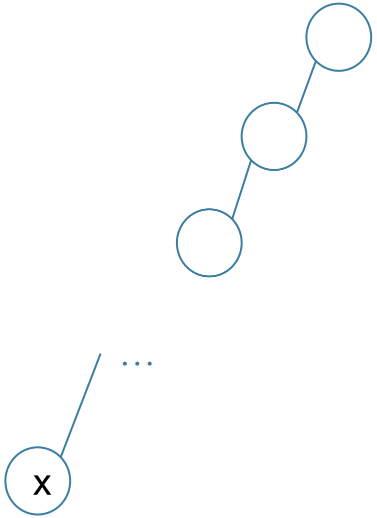
2 Fill in the Blanks

5. Given a min-heap with $2^n - 1$ elements, for an element to be on the bottommost level it must be less than ____ element(s) and greater than _____ element(s).

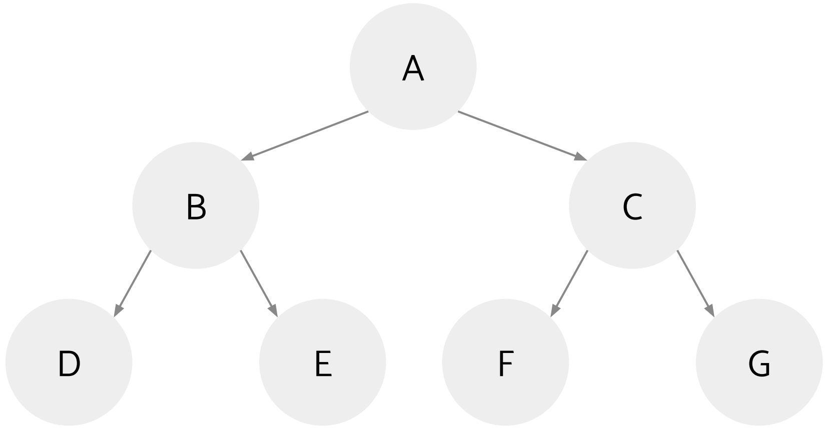
2 Fill in the Blanks

5. Given a min-heap with 2^n - 1 elements, for an element to be on the bottommost level it must be less than 0 element(s) and greater than N - 1 element(s).

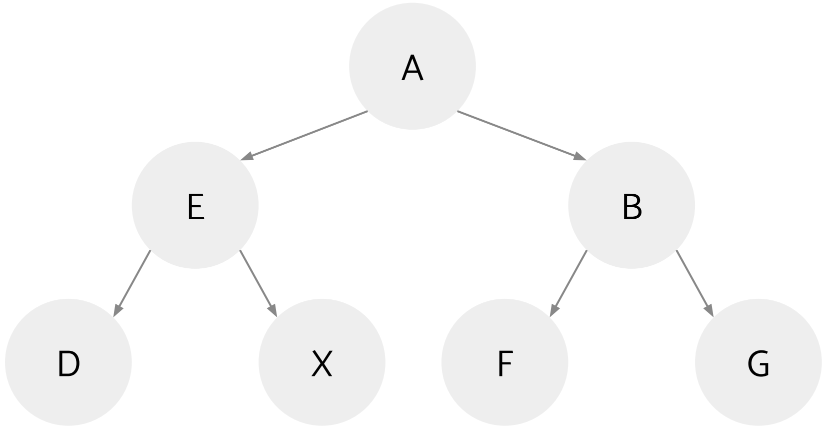
larger than all direct ancestors



3a Heap Mystery

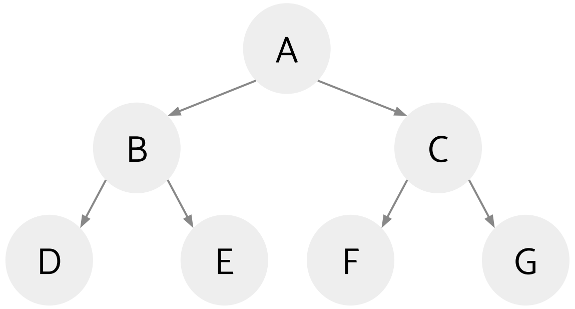


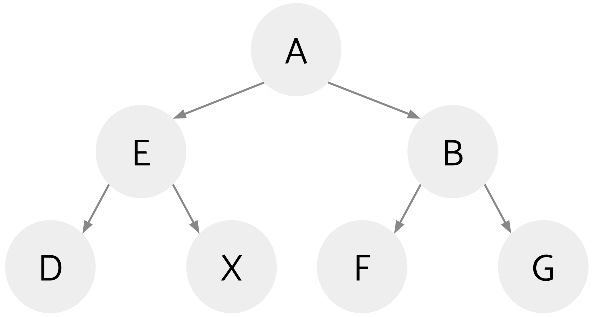
Initial State
[-, A, B, C, D, E, F, G]



Final State:
[-, A, E, B, D, X, F, G]

3a Heap Mystery





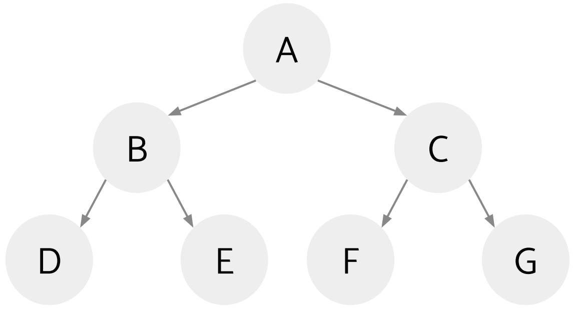
Sequence of calls:

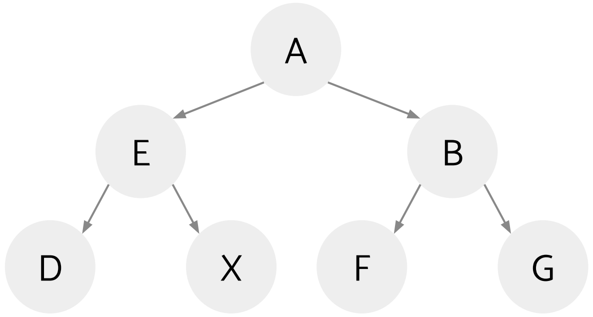
1. `removeMin()`
2. ____________
3. ____________
4. ____________

Differences in state:

- C was removed: `removeMin()`
- X was added: insert(X)
- A was removed by first call to `removeMin()` and added back: `insert(A)`

3a Heap Mystery





`insert(A)` must be after all `removeMin()` – otherwise would remove A again

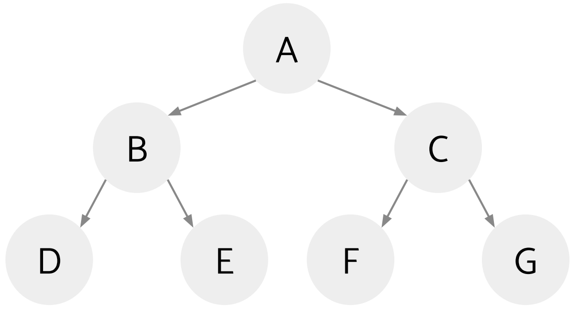
Sequence of calls:

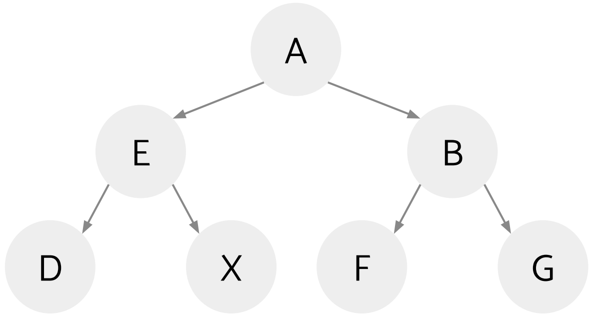
1. `removeMin()`
2. `removeMin()` / `insert(X)`
3. `removeMin()` / `insert(X)`
4. `insert(A)`

Differences in state:

- C was removed: `removeMin()`
- X was added: insert(X)
- A was removed by first call to `removeMin()` and added back: `insert(A)`

3a Heap Mystery





`insert(X)` must be before removeMin, since it bubbles up then down - that's the only way it's able to change sides.

Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. `removeMin()`
4. `insert(A)`

Differences in state:

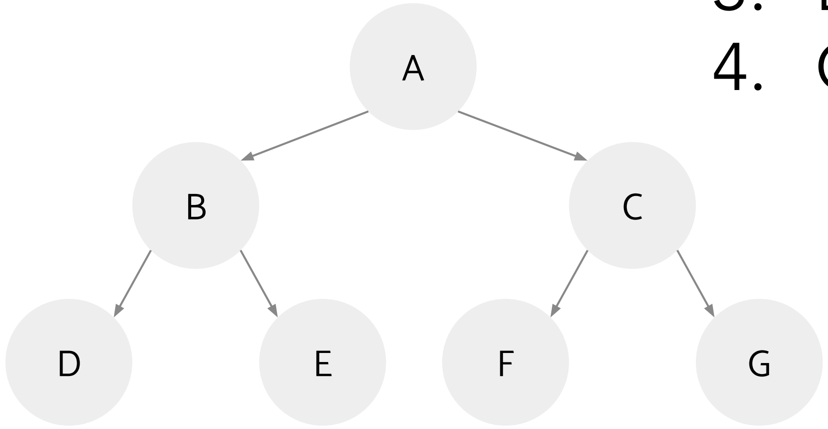
- C was removed: `removeMin()`
- X was added: insert(X)
- A was removed by first call to `removeMin()` and added back: `insert(A)`

3b Heap Mystery

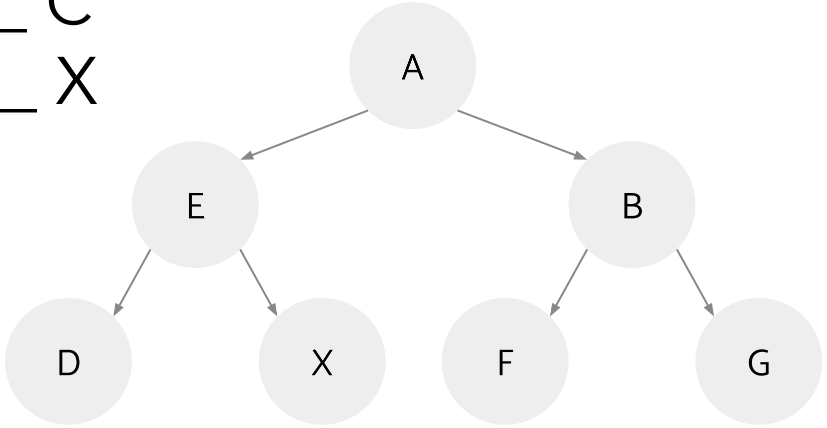
1. X ____ D
2. X ____ C
3. B ____ C
4. G ____ X

Sequence of calls:

1. `removeMin() -> A`
2. `insert(X)`
3. `removeMin() -> C`
4. `insert(A)`



Initial State
[-, A, B, C, D, E, F, G]



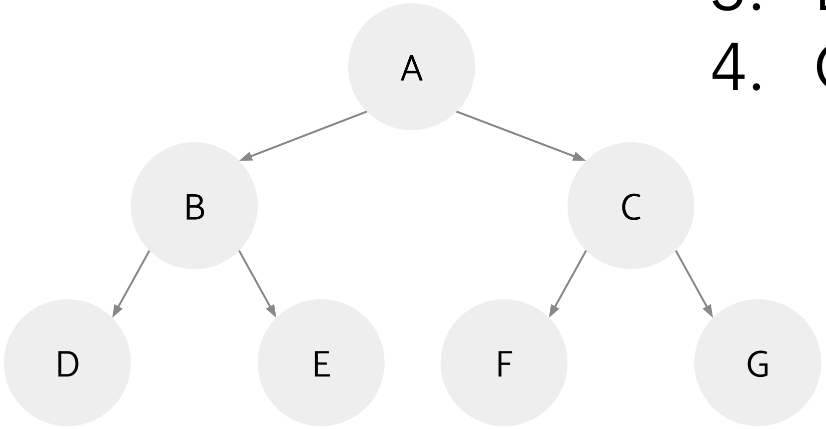
Final State:
[-, A, E, B, D, X, F, G]

3b Heap Mystery

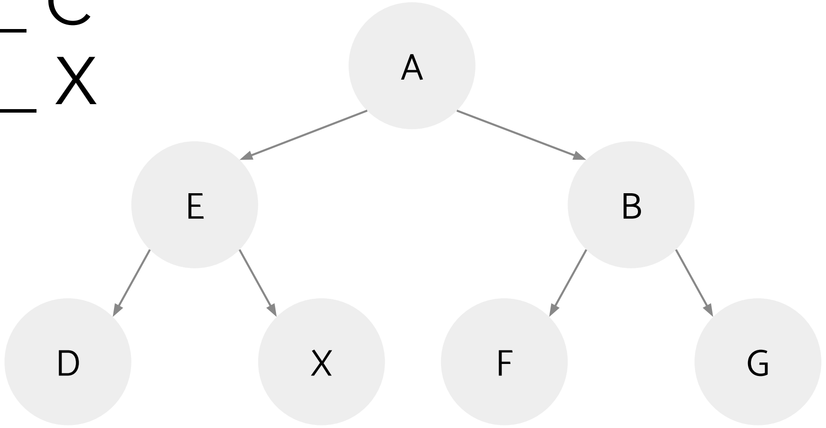
1. X ? D
2. X _____ C
3. B _____ C
4. G _____ X

Sequence of calls:

1. `removeMin() -> A`
2. `insert(X)`
3. `removeMin() -> C`
4. `insert(A)`



Initial State
[-, A, B, C, D, E, F, G]



Final State:
[-, A, E, B, D, X, F, G]

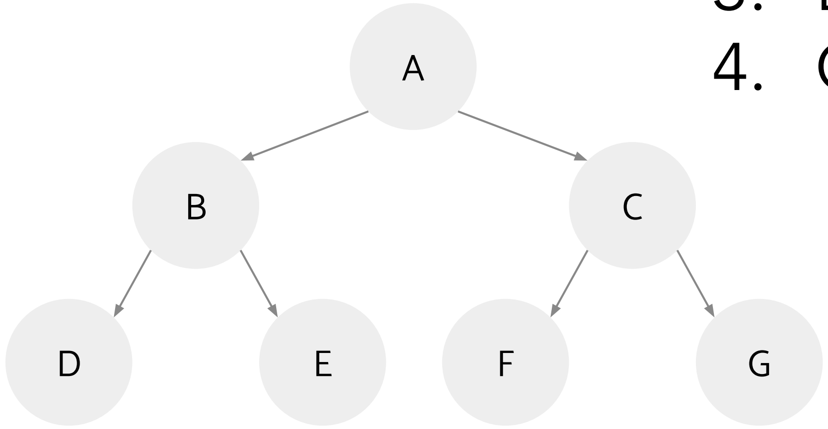
X is never compared to D

3b Heap Mystery

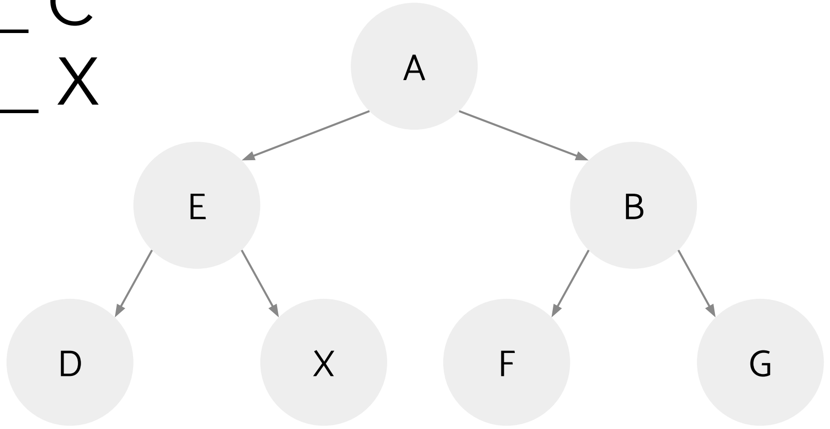
1. X ? D
2. X > C
3. B ____ C
4. G ____ X

Sequence of calls:

1. `removeMin()-> A`
2. `insert(X)`
3. `removeMin()-> C`
4. `insert(A)`



Initial State
[-, A, B, C, D, E, F, G]



Final State:
[-, A, E, B, D, X, F, G]

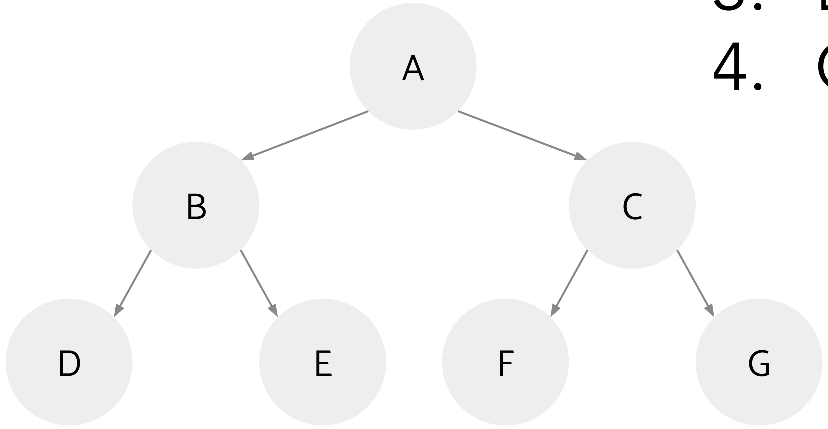
At step 3, C is less than any other element in the heap at that time.

3b Heap Mystery

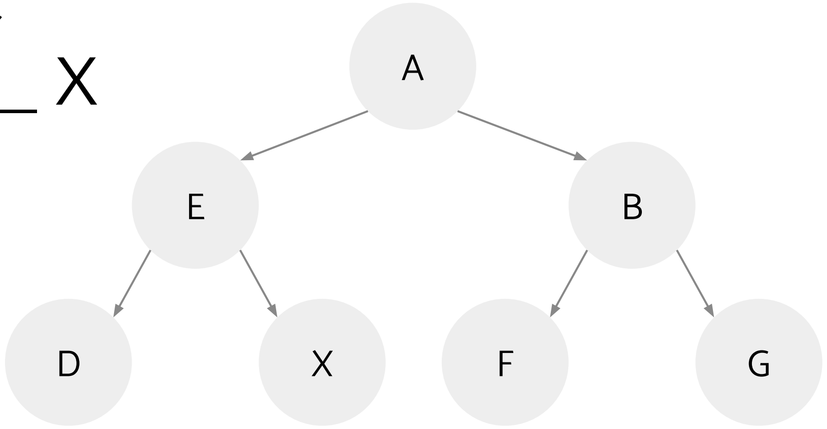
1. X ? D
2. X > C
3. B > C
4. G ____ X

Sequence of calls:

1. `removeMin() -> A`
2. `insert(X)`
3. `removeMin() -> C`
4. `insert(A)`



Initial State
[-, A, B, C, D, E, F, G]



Final State:
[-, A, E, B, D, X, F, G]

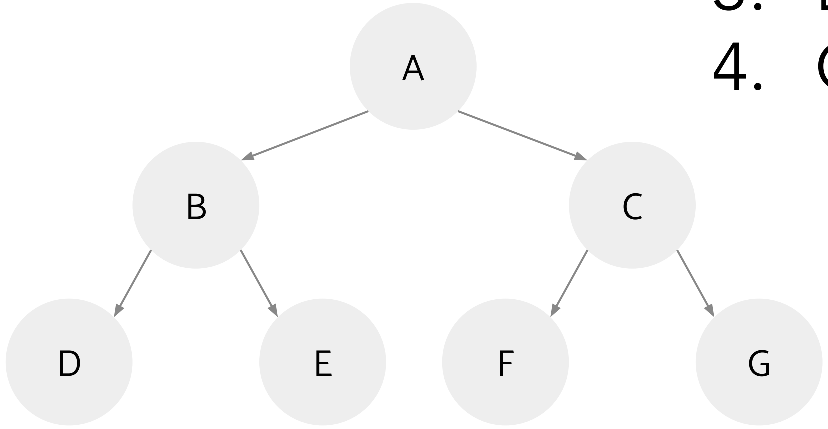
At step 3, C is less than any other element in the heap at that time.

3b Heap Mystery

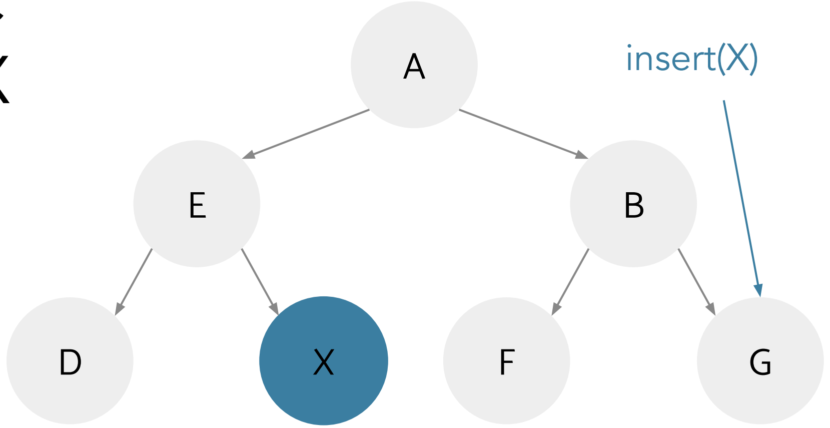
1. X ? D
2. X > C
3. B > C
4. G < X

Sequence of calls:

```
1.  removeMin()-> A
2.  insert(X)
3.  removeMin()-> C
4.  insert(A)
```



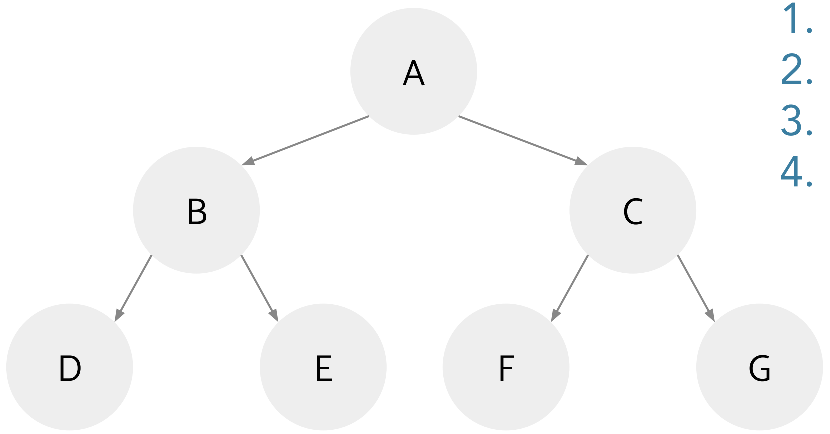
Initial State
[-, A, B, C, D, E, F, G]



Final State:
[-, A, E, B, D, X, F, G]

X must stay under G at step 2, so that we can swap it to the top at step 3 during the remove, where it then bubbles down.

3b Heap Mystery

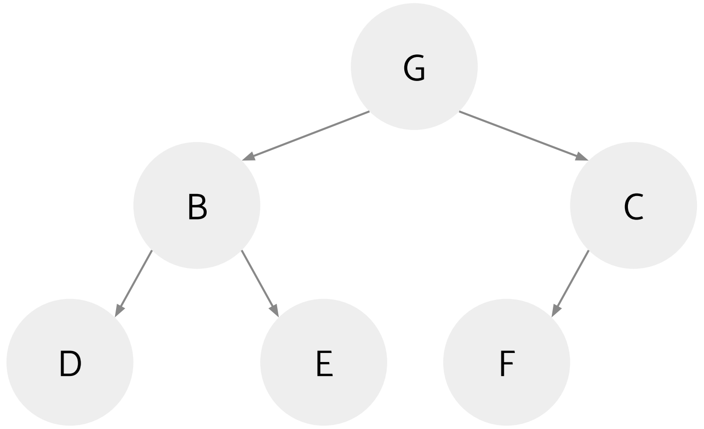


Initial State
[-, A, B, C, D, E, F, G]

Sequence of calls:

1. **`removeMin()`**
2. `insert(X)`
3. `removeMin()`
4. `insert(A)`

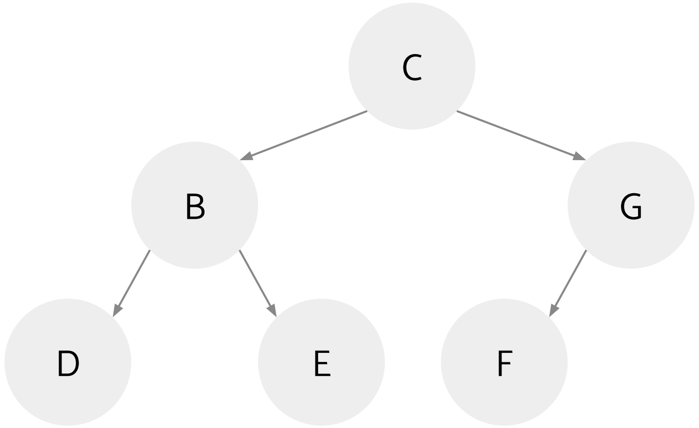
3b Heap Mystery



Sequence of calls:

1. **`removeMin()`**
2. `insert(X)`
3. `removeMin()`
4. `insert(A)`

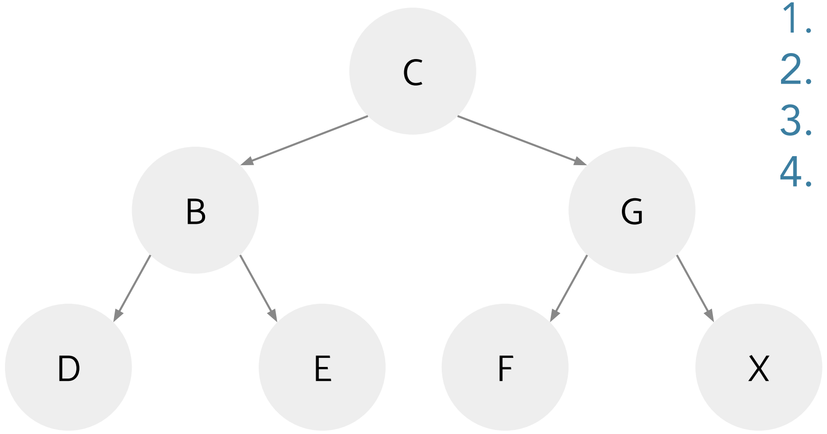
3b Heap Mystery



Sequence of calls:

1. **`removeMin()`**
2. `insert(X)`
3. `removeMin()`
4. `insert(A)`

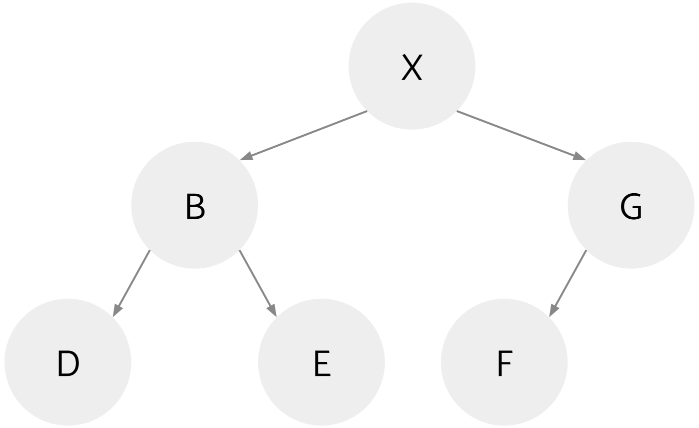
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. **`insert(X)`**
3. `removeMin()`
4. `insert(A)`

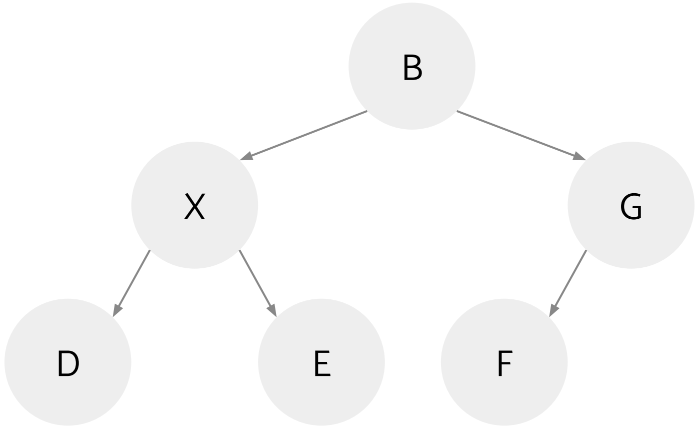
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. **`removeMin()`**
4. `insert(A)`

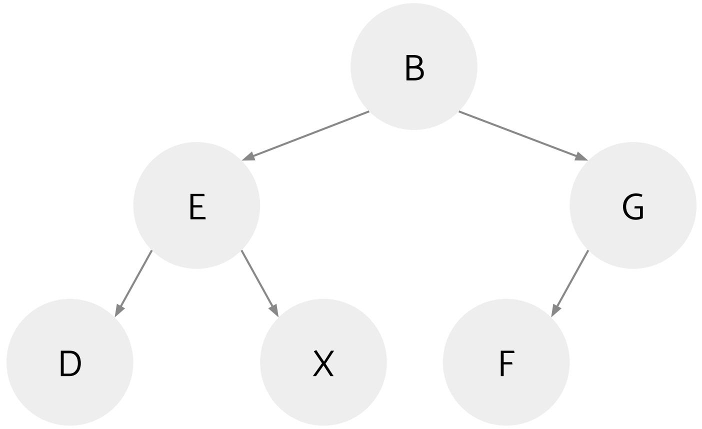
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. **`removeMin()`**
4. `insert(A)`

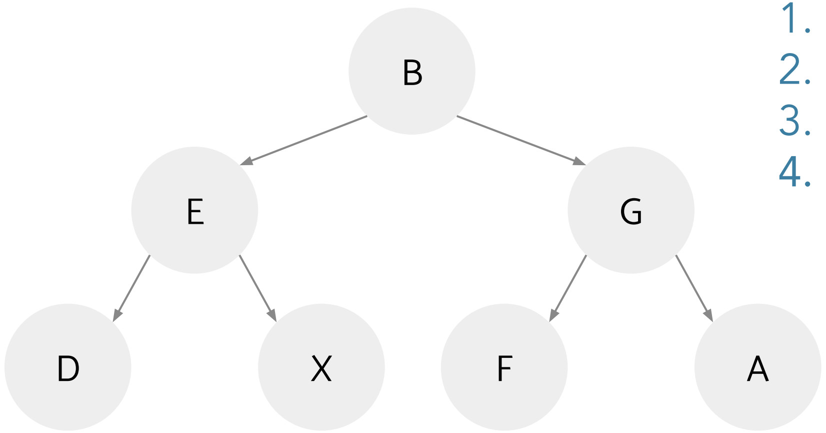
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. **`removeMin()`**
4. `insert(A)`

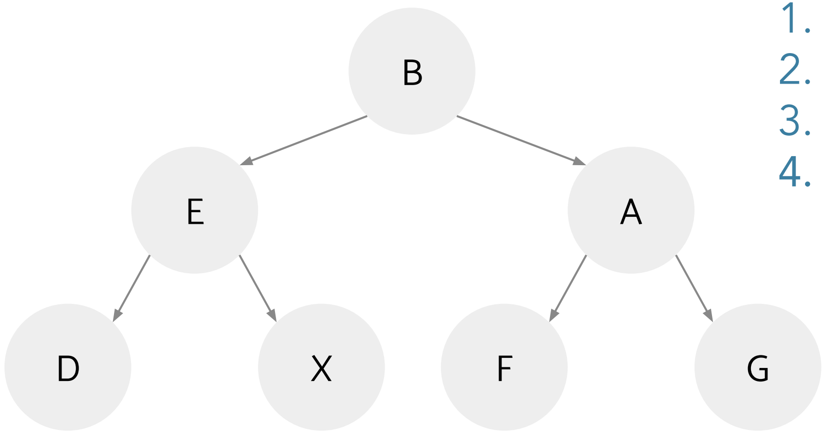
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. `removeMin()`
4. **`insert(A)`**

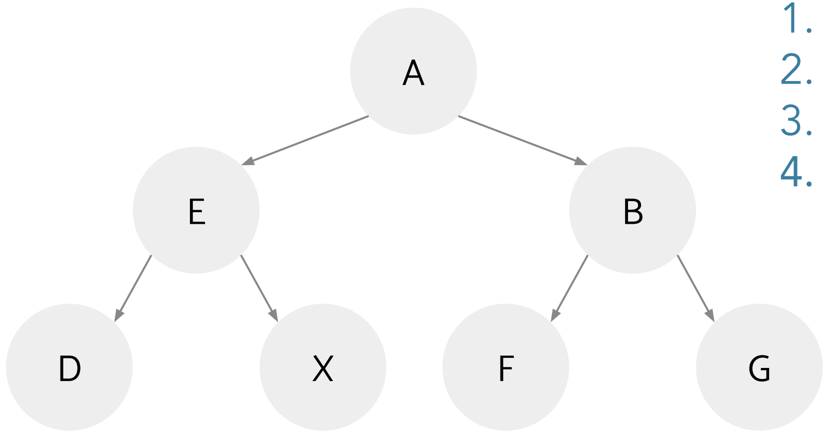
3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. `removeMin()`
4. **`insert(A)`**

3b Heap Mystery



Sequence of calls:

1. `removeMin()`
2. `insert(X)`
3. `removeMin()`
4. **`insert(A)`**

Final State:
[-, A, E, B, D, X, F, G]